

Mean time exit and isoperimetric inequalities for minimal submanifolds of $N \times \mathbb{R}$

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Abstract

Based on Markvorsen and Palmer's work on mean time exit and isoperimetric inequalities we establish slightly better isoperimetric inequalities and mean time exit estimates for minimal submanifolds of $N \times \mathbb{R}$. We also prove isoperimetric inequalities for submanifolds of Hadamard spaces with tamed second fundamental form.

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1 Introduction

The study of minimal surfaces in product spaces $N \times \mathbb{R}$, where N is a complete surface, started with H. Rosenberg in [22] and it has shown to be a rich and interesting theory, yielding a wealth of examples and results, [10], [11], [12], [17], [18], [23]. It also lead to the study of constant mean curvature surfaces in product spaces, [1], [3], [6], [9], [13], [19], [20]. The classical theory of minimal surfaces in \mathbb{R}^3 guides the search in this new theory, that depending on the geometry of N , yields very different results from their counterparts in the classical theory. In this spirit, based on the ideas of Markvorsen-Palmer, we study isoperimetric inequalities for minimal submanifolds of $N \times \mathbb{R}$, where N is a complete Riemannian n -manifold with sectional curvature $K_N \leq b$. Markvorsen and Palmer in [16] and [21] proved isoperimetric inequalities for extrinsic geodesic balls of proper minimal submanifolds of Riemannian manifolds with sectional curvature bounded above. To be precise, let $\varphi : M \hookrightarrow W$ be a proper minimal immersion of an m -dimensional manifold M into a Riemannian n -manifold W with sectional curvature $K_W \leq b$ and let $B_W(R)$ be a geodesic ball of W centered at a point $p = \varphi(q)$ with radius $R \leq \min\{\text{inj}_W(p), \pi/2\sqrt{b}\}$, where $\pi/2\sqrt{b} = \infty$ if $b \leq 0$ and $\text{inj}_W(p)$ is the injectivity radius at p . The extrinsic geodesic ball of radius R centered at p , denoted by $D(R)$, is defined to be the

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connected component of $\varphi(M) \cap B_W(R)$ containing p . The isoperimetric inequalities proved by Markvorsen and Palmer are the following.

Theorem 1.1 (Markvorsen-Palmer, [16], [21])

i. If $b \leq 0$ then

$$\frac{\text{vol}_{m-1}(\partial D(R))}{\text{vol}_m(D(R))} \geq \frac{\text{vol}_{m-1}(\partial B_{\mathbb{N}^m(b)}(R))}{\text{vol}_m(B_{\mathbb{N}^m(b)}(R))} \quad (1.1)$$

ii. If $b > 0$ then

$$\frac{\text{vol}_{m-1}(\partial D(R))}{\text{vol}_m(D(R))} \geq m \frac{C_b}{S_b}(R). \quad (1.2)$$

Where $(C_b/S_b)(R)$ is the (constant) mean curvature of the geodesic sphere $\partial B_{\mathbb{N}^m(b)}(R)$ of radius R in $\mathbb{N}^m(b)$ and the functions S_b and C_b are defined in (1.4). Moreover, equality in item i. implies that $D(R)$ is a minimal cone in W .

We consider a minimal immersion $\varphi : M \hookrightarrow N \times \mathbb{R}$ of m -dimensional manifold M into the product space $N \times \mathbb{R}$, where N is a complete Riemannian manifold with sectional curvature $K_N \leq b$. Let $K \subset \varphi(M)$ be a connected compact set and let $r_K = \text{rad}(\pi_1(K))$ be the radius of the set $\pi_1(K)$, where $\pi_1 : N \times \mathbb{R} \rightarrow N$ is the projection on the first factor. Denote by $p_K \in N$ the barycenter of $\pi_1(K)$ and suppose that $r_K < \min\{\text{inj}_N(p_K), \pi/2\sqrt{b}\}$. We prove a slightly better isoperimetric inequality when $b < 0$.

Theorem 1.2 If $b \leq 0$ then

$$\frac{\text{vol}_{m-1}(\partial K)}{\text{vol}_m(K)} \geq \frac{\text{vol}_{m-2}(\partial B_{\mathbb{N}^{m-1}(b)}(r_K))}{\text{vol}_{m-1}(B_{\mathbb{N}^{m-1}(b)}(r_K))}. \quad (1.3)$$

Remark 1.3 This isoperimetric inequality is sharp if we consider arbitrary compact sets K . Consider the totally geodesic embedding $\varphi : \mathbb{H}^{m-1} \times \mathbb{R} \hookrightarrow \mathbb{H}^n(-1) \times \mathbb{R}$ given by $\varphi(x, t) = (x, t)$ and a family of compact sets $K_i = B_{\mathbb{H}^{m-1}}(R) \times [-i, i]$, $i = 1, 2, \dots$. We then have that

$$\frac{\text{vol}_{m-1}(\partial K_i)}{\text{vol}_m(K_i)} = \frac{\text{vol}_{m-2}(\partial B_{\mathbb{H}^{m-1}}(R))}{\text{vol}_{m-1}(B_{\mathbb{H}^{m-1}}(R))} + \frac{1}{i} \rightarrow \frac{\text{vol}_{m-2}(\partial B_{\mathbb{H}^{m-1}}(R))}{\text{vol}_{m-1}(B_{\mathbb{H}^{m-1}}(R))}.$$

On the other hand, if we consider extrinsic geodesic balls $D(R)$ of $M \hookrightarrow N \times \mathbb{R}$, $K_N \leq -1$, then Markvorsen-Palmer's estimate (1.1) is better if $m = 2$. Since

$$\frac{\text{vol}_1(\partial D(R))}{\text{vol}_2(D(R))} \geq \frac{\text{vol}_1(\partial B_{\mathbb{R}^2}(R))}{\text{vol}_2(B_{\mathbb{R}^2}(R))} = \frac{2}{R} > \frac{1}{R} = \frac{\text{vol}_0(\partial B_{\mathbb{H}^1}(R))}{\text{vol}_1(B_{\mathbb{H}^1}(R))}.$$

For $m \geq 3$ there exists R_m such that if $R \geq R_m$ then we have

$$\frac{\text{vol}_{m-1}(\partial D(R))}{\text{vol}_m(D(R))} \geq \frac{\text{vol}_{m-2}\partial B_{\mathbb{H}^{m-1}(-1)}(R)}{\text{vol}_{m-1}B_{\mathbb{H}^{m-1}(-1)}(R)} = \frac{\sinh(R)^{m-2}}{\int_0^R \sinh(s)^{m-2} ds} \geq \frac{m}{R} = \frac{\text{vol}_{m-1}(\partial B_{\mathbb{R}^m}(R))}{\text{vol}_m(B_{\mathbb{R}^m}(R))}.$$

In fact, a rough estimate gives

$$\frac{\sinh(R)^{m-2}}{\int_0^R \sinh(s)^{m-2} ds} \geq (m-2) \cdot \frac{(e^R - 1)^{m-2}}{e^{(m-2)R} - 1}$$

Just let R_m be such that

$$(m-2) \cdot \frac{(e^{R_m} - 1)^{m-2}}{e^{(m-2)R_m} - 1} = \frac{m}{R_m}.$$

Our next result gives upper bounds for the isoperimetric quotients for extrinsic geodesic balls of submanifolds with tamed second fundamental form in Hadamard spaces with bounded sectional curvature.

Let $\varphi : M \hookrightarrow N$ be an isometric immersion of a complete Riemannian m -manifold M into a Hadamard n -manifold N with sectional curvature bounded above $K_N \leq b \leq 0$. Fix a point $x_0 \in M$ and let $\rho_M(x) = \text{dist}_M(x_0, x)$ be the distance function on M to x_0 .

Let $\{C_i\}_{i=1}^\infty$ be an exhaustion sequence of M by compact sets with $x_0 \in C_1$ and define a non-increasing sequence $a_1 \geq a_2 \geq \dots \geq 0$ by

$$a_i = \sup \left\{ \frac{S_b(\rho_M(x))}{C_b(\rho_M(x))} \cdot \|\alpha(x)\|, x \in M \setminus C_i \right\},$$

where

$$S_b(t) = \begin{cases} \frac{1}{\sqrt{-b}} \sinh(\sqrt{-b} t), & \text{if } b < 0 \\ t, & \text{if } b = 0, \\ \frac{1}{\sqrt{b}} \sin(\sqrt{b} t), & \text{if } b > 0 \text{ and } t < \pi/2\sqrt{b} \end{cases} \quad (1.4)$$

$C_b(t) = S'_b(t)$ and $\alpha(x)$ is the second fundamental form of $\varphi(M)$ at $\varphi(x)$. It is clear that the limit $a(M) = \lim_{i \rightarrow \infty} a_i \in [0, \infty]$ does not depend on the exhaustion sequence $\{C_i\}_{i=1}^\infty$ nor on the base point x_0 .

Definition 1.4 *An immersion $\varphi : M \hookrightarrow N$ of a complete Riemannian m -manifold M into a Hadamard n -manifold N with sectional curvature $K_N \leq b \leq 0$ has tamed second fundamental form if $a(M) < 1$.*

Submanifolds of \mathbb{R}^n with tamed second fundamental form were studied by the authors and L. Jorge in [4] where we showed that complete submanifolds with tamed fundamental form of the \mathbb{R}^n are proper and has finite topology. Silvana Costa [7] extended this result to submanifolds of Hadamard manifolds with tamed second fundamental form. Here we give upper bounds for the isoperimetric quotients of extrinsic geodesic balls in Hadamard manifolds. We prove the following theorem.

Theorem 1.5 *Let $\varphi : M \hookrightarrow N$ be a complete immersed m -submanifold M with tamed second fundamental form of an n -dimensional Hadamard manifold N with bounded sectional curvature $b_1 \leq K_N \leq b_2 \leq 0$. For a given $c \in (a(M), 1)$ there exists positive constants $r_0 = r_0(b_2, c)$, $B = B(b_2, c) < 1$ such that for extrinsic geodesic balls $D(R)$ with radius $R \geq r_0$ we have*

$$\frac{\text{vol}_{m-1}(\partial D(R))}{\text{vol}_m(D(R))} \leq \frac{1 + \sqrt{-b_1} \cdot R \cdot \coth(\sqrt{-b_1} \cdot R) + \Lambda}{R \cdot \sqrt{1 - B^2}} \quad (1.5)$$

Where Λ is a constant depending on c, r_0, R, b_2 and $\sup_{B_N(r_0)} |H|$.

2 Proof of the results

2.1 Basic formulas

Let $\varphi : M \hookrightarrow W$ be an isometric immersion M and W are Riemannian manifolds. Consider a smooth function $g : W \rightarrow \mathbb{R}$ and the composition $f = g \circ \varphi : M \rightarrow \mathbb{R}$. Let ∇ and $\bar{\nabla}$ be the Riemannian connections on M and W respectively, $\alpha(q)(X, Y)$ and $\text{Hess } f(q)(X, Y)$ be respectively the second fundamental form of the immersion φ and the Hessian of f at $q \in M$, $X, Y \in T_p M$. Identifying X with $d\varphi(X)$ we have at $q \in M$ and for every $X \in T_q M$ that

$$\text{Hess } f(q)(X, Y) = \text{Hess } g(\varphi(q))(X, Y) + \langle \text{grad } g, \alpha(X, Y) \rangle_{\varphi(q)}. \quad (2.1)$$

Taking the trace in (2.1), with respect to an orthonormal basis $\{e_1, \dots, e_m\}$ for $T_q M$, we have that

$$\begin{aligned} \Delta f(q) &= \sum_{i=1}^m \text{Hess } f(q)(e_i, e_i) \\ &= \sum_{i=1}^m \text{Hess } g(\varphi(q))(e_i, e_i) + \langle \text{grad } g, \sum_{i=1}^m \alpha(e_i, e_i) \rangle. \end{aligned} \quad (2.2)$$

The formulas (2.1) and (2.2) are well known in the literature, see [14]. Another important tool is the Hessian Comparison Theorem, see [24].

Theorem 2.1 (Hessian Comparison Theorem) *Let W be a complete Riemannian n -manifold and $y_0, y_1 \in W$. Let $\gamma : [0, \rho_W(y_1)] \rightarrow W$ be a minimizing geodesic joining y_0 and y_1 where ρ_W is the distance function to y_0 on W . Let K_γ be the sectional curvatures of W along γ and let $c = \inf K_\gamma$ and $b = \sup K_\gamma$. The Hessian of ρ_W at $y = \gamma(\rho_W(y))$ for any $X \in T_y W$, $X \perp \gamma'(\rho_W(y))$, satisfies*

$$\frac{C_c}{S_c}(\rho_W(y)) \cdot \|X\|^2 \geq \text{Hess } \rho_W(y)(X, X) \geq \frac{C_b}{S_b}(\rho_W(y)) \cdot \|X\|^2, \quad (2.3)$$

whereas $\text{Hess } \rho_W(y)(\gamma', \gamma') = 0$.

2.2 Mean time exit from minimal submanifolds of $N \times \mathbb{R}$

Let $\varphi : M \hookrightarrow W$ be a complete, minimal, properly immersed m -submanifold of a complete Riemannian manifold W with sectional curvature $K_W \leq b$. Let $D(R)$ be an extrinsic geodesic ball centered at $p = \varphi(q)$ with radius R and $\rho_w(x) = \text{dist}_W(p, x)$. Let $E(x)$ be the mean time of the first exit from $D(R)$ of a particle in Brownian motion starting at $x \in D(R)$ and denote by $E_b^m(\tilde{x}) = E_b^m(|\tilde{x}|)$ the mean time of the first exit from $B_{\mathbb{N}^m(b)}(R)$ of a particle in Brownian motion starting at $\tilde{x} \in B_{\mathbb{N}^m(b)}(R)$, $|\tilde{x}| = \text{dist}_{\mathbb{N}^m(b)}(0, \tilde{x})$. Markvorsen, [15] proved the following theorem.

Theorem 2.2 (Markvorsen's mean time exit comparison theorem)

- i. If the sectional curvature $b \geq K_W \geq \kappa \geq 0$ then $E(x) \geq E_\kappa^m(\rho_w(x))$.
- ii. If the sectional curvature $0 \geq b \geq K_W$ then $E(x) \leq E_b^m(\rho_w(x))$.

We have a version of Markvorsen's mean time exit comparison theorem for compact sets of minimal submanifolds of $N \times \mathbb{R}$. Let $K \subset \varphi(M)$ be compact set in a minimal m -submanifold of $N \times \mathbb{R}$, where N is a Riemannian n -manifold with sectional curvature $K_N \leq b$. Let r_K and p_K be respectively the radius and barycenter of $\pi_1(K)$. Suppose that $r_K < \min\{\text{inj}_N(p_K), \pi/2\sqrt{b}\}$. Denote by $E(x)$ the mean time of the first exit from K of a particle in Brownian motion starting at $x \in K$ and by $E_b(\tilde{x}) = E_b^{m-1}(|\tilde{x}|)$ the mean time of the first exit from $B_{\mathbb{N}^{m-1}(b)}(r_K)$ of a particle in Brownian motion starting at $\tilde{x} \in B_{\mathbb{N}^{m-1}(b)}(r_K)$, $|\tilde{x}| = \text{dist}_{\mathbb{N}^{m-1}(b)}(0, \tilde{x})$.

We prove the following comparison theorem.

Theorem 2.3

- i. If $K_N \leq b \leq 0$ then $E(x) \leq E_b(\rho_N(\pi_1(x)))$.
- ii. If $K_N \geq \kappa \geq 0$, suppose that the immersion φ is proper and $K = \varphi(M) \cap (B_N(R) \times \mathbb{R})$ is a compact set, then $E(x) \geq E_\kappa(\rho_N(\pi_1(x)))$. Where $\rho_N(\pi_1(x)) = \text{dist}_N(p_K, \pi_1(x))$.

Remark 2.4 The statement of this theorem is somewhat surprising. For instance, consider the totally geodesic embedding $\varphi : \mathbb{H}^{m-1} \times \mathbb{R} \hookrightarrow \mathbb{H}^n(-1) \times \mathbb{R}$ given by $\varphi(x, t) = (x, t)$ and $K = B_{\mathbb{H}^{m-1}}(R) \times [-L, L]$. It does not matter how large is L , the mean time exit of K can not exceed $E_{-1}(R)$. The particle in Brownian motion can not move upward for too long. It is drifted horizontally to the boundary.

2.3 Proof of Theorem 2.3

We denoted by $E(x)$ the mean time of the first exit from K of a particle in Brownian motion starting at x and by $E_b(\tilde{x})$ the mean time from the first exit of the geodesic ball $B_{\mathbb{N}^{m-1}(b)}(r_K)$ of a particle in Brownian motion starting at \tilde{x} . A remark from Dynkin [8] vol 2, p.51 states that the functions E and E_b satisfies the Dirichlet boundary problem.

$$\begin{cases} \Delta_K E = -1 & \text{in } K \\ E = 0 & \text{on } \partial K \end{cases} \quad \& \quad \begin{cases} \Delta_{\mathbb{N}^{m-1}(b)} E_b = -1 & \text{in } B_{\mathbb{N}^{m-1}(b)}(r_K) \\ E_b = 0 & \text{on } \partial B_{\mathbb{N}^{m-1}(b)}(r_K) \end{cases} \quad (2.4)$$

It is known that E_b is a radial function $E_b(\tilde{x}) = E_b(|\tilde{x}|)$, $|\tilde{x}| = \text{dist}_{\mathbb{N}^{m-1}(b)}(0, \tilde{x})$. Let \bar{E}_b be the transplant of E_b to $B_N(r_K) \times \mathbb{R}$ defined by $\bar{E}_b(x) = E_b \circ \rho_N \circ \pi_1(x)$, where $\pi_1 : N \times \mathbb{R} \rightarrow N$ is the projection on the first factor. We have that $\bar{E}_b|_K = \bar{E}_b \circ \varphi$. Following Markvorsen [15] we define $F_b : [0, \infty) \rightarrow [0, \infty)$ by

$$F_b(t) = \begin{cases} \frac{1}{b}(1 - \cos(\sqrt{b} \cdot t)) & \text{if } b > 0 \\ \frac{t^2}{2} & \text{if } b = 0 \\ \frac{1}{b}(1 - \cosh(\sqrt{-b} \cdot t)) & \text{if } b < 0 \end{cases} \quad (2.5)$$

Observe that F_b satisfies $F_b''(t) - (C_b/S_b)(t)F_b'(t) = 0$ for all $t \geq 0$.

Let $s = F_b(\rho_N \circ \pi_1)$ and define $\bar{\mathbf{E}}_b(s)$ by $\bar{\mathbf{E}}_b(s(x)) = \bar{E}_b(x)$. Computing $\triangle_K \bar{E}_b \circ \varphi$ at any point $x \in B_N(R)$ we obtain, (see 2.2)

$$\begin{aligned} \triangle_K \bar{E}_b \circ \varphi(x) &= \sum_{i=1}^m \text{Hess}_{(N \times \mathbb{R})} \bar{E}_b(y)(X_i, X_i) \\ &= \sum_{i=1}^m \text{Hess}_{(N \times \mathbb{R})} E_b \circ \rho_N \circ \pi_1(y)(X_i, X_i) \\ &= \sum_{i=1}^m \text{Hess}_N(E_b \circ \rho_N \circ \pi_1)(y)(X_i, X_i) \\ &= \sum_{i=1}^m \text{Hess}_N \bar{\mathbf{E}}_b(s(y))(X_i, X_i) \\ &= \sum_{i=1}^m [\bar{\mathbf{E}}_b''(s(y)) \langle \text{grad } s, X_i \rangle^2 + \bar{\mathbf{E}}_b'(s(y)) \text{Hess}_N s(y)(X_i, X_i)] \end{aligned} \quad (2.6)$$

Where $\{X_i\}$ is an orthonormal basis for $T_y \varphi(M)$, $y = \varphi(x)$. Let $\{\partial/\partial \rho_N, \partial/\partial \theta_1, \dots, \partial/\partial \theta_n\}$ be an orthonormal basis for $T_{\pi_1(y)}N$ from polar coordinates and $\partial/\partial t$ is the a tangent to the \mathbb{R} factor. We choose $\{X_i\}$ in the following way.

$$X_i = \alpha_i \cdot \partial/\partial \rho_N + \beta_i \cdot \partial/\partial t + \sum_{j=1}^{n-1} \gamma_j^i \cdot \partial/\partial \theta_j \quad (2.7)$$

$$\alpha_i^2 + \beta_i^2 + \sum_{j=1}^{n-1} (\gamma_j^i)^2 = 1 \quad . \quad (2.8)$$

We compute $\sum_{i=1}^m \text{Hess}_N s(X_i, X_i)$ taking in account the Hessian Comparison Theorem and the fact $F_b''(t) - F_b'(t)(C_b/S_b)(t) = 0$.

$$\begin{aligned}
\sum_{i=1}^m \text{Hess}_N s(X_i, X_i) &= F_b''(\rho_N) \sum_{i=1}^m \langle \text{grad } \rho_N, X_i \rangle^2 + F_b'(\rho_N) \sum_{i=1}^m \text{Hess}_N \rho_N(X_i, X_i) \\
&= F_b''(\rho_N) \sum_{i=1}^m \alpha_i^2 + F_b'(\rho_N) \sum_{i=1}^m \sum_{j=1}^{n-1} (\gamma_j^i)^2 \text{Hess}_N \rho_N(\partial/\partial\theta_j, \partial/\partial\theta_j) \quad (2.9) \\
&\geq F_b''(\rho_N) \sum_{i=1}^m \alpha_i^2 + F_b'(\rho_N) \frac{C_b}{S_b}(\rho_N) \sum_{i=1}^m (1 - \alpha_i^2 - \beta_i^2) \\
&= \left(F_b''(\rho_N) - F_b'(\rho_N) \frac{C_b}{S_b}(\rho_N) \right) \sum_{i=1}^m \alpha_i^2 + F_b'(\rho_N) \frac{C_b}{S_b}(\rho_N) (m - \sum_{i=1}^m \beta_i^2) \\
&\geq (m-1) F_b'(\rho_N) \frac{C_b}{S_b}(\rho_N)
\end{aligned}$$

Thus

$$\sum_{i=1}^m \text{Hess}_N s(X_i, X_i) \geq (m-1) \cdot F_b'(\rho_N) \frac{C_b}{S_b}(\rho_N). \quad (2.10)$$

Recall that the Laplacian of the canonical metric $dt^2 + S_b^2(t)d\theta^2$ of the space form $\mathbb{N}^{m-1}(b)$ is given by $\Delta_{\mathbb{N}^{m-1}(b)} = \partial^2/\partial t^2 + (m-2)(C_b/S_b)\partial/\partial t + (1/S_b^2(t))\Delta_{\mathbb{S}^{m-2}}$.

Therefore

$$\Delta_{\mathbb{N}^{m-1}(b)} s = \Delta_{\mathbb{N}^{m-1}(b)} F_b(\rho_N) = F_b''(\rho_N) + (m-2) \frac{C_b}{S_b} F_b'(\rho_N) = (m-1) \frac{C_b}{S_b} F_b'(\rho_N).$$

In [15], Proposition 4, Markvorsen proved that $\bar{\mathbf{E}}'_b(s) < 0$ for all b and $\bar{\mathbf{E}}''_b(s) > 0$ if $b < 0$, $\bar{\mathbf{E}}''_b(s) = 0$ if $b = 0$ and $\bar{\mathbf{E}}'_b(s) < 0$ if $b > 0$.

Therefore from (2.6) we have

$$\begin{aligned}
\Delta_K \bar{E}_b \circ \varphi(x) &= \sum_{i=1}^m [\bar{\mathbf{E}}''_b(s) \langle \text{grad } s, X_i \rangle^2 + \bar{\mathbf{E}}'_b(s) \text{Hess}_N s(X_i, X_i)] \\
&\leq \bar{\mathbf{E}}''_b(s) |\text{grad}_N s|^2 + \bar{\mathbf{E}}'_b(s) \cdot (m-1) \cdot F_b'(\rho_N) \frac{C_b}{S_b}(\rho_N) \\
&= \bar{\mathbf{E}}''_b(s) |\text{grad}_{\mathbb{N}^{m-1}(b)} s|^2 + \bar{\mathbf{E}}'_b(s) \Delta_{\mathbb{N}^{m-1}(b)} s \\
&= \Delta_{\mathbb{N}^{m-1}(b)} \bar{\mathbf{E}}_b = -1 = \Delta_K E
\end{aligned} \quad (2.11)$$

Then $\Delta_K(\bar{E}_b - E) \leq 0$ with $(\bar{E}_b - E)|_{\partial K} = \bar{E}_b|_{\partial K} \geq 0$. Thus $\bar{E}_b \geq E$ in K .

If $K_N \geq \kappa > 0$ then

$$\sum_{i=1}^m \text{Hess}_N s(X_i, X_i) \leq (m-1) \cdot F'_b(\rho_N) \frac{C_b}{S_b}(\rho_N). \quad (2.12)$$

and $\bar{E}''_\kappa(s) < 0$. The same reasoning as before shows that $\Delta_K(\bar{E}_\kappa - E) \geq 0$ and this is valid for any compact K . But our compact set in consideration is $K = \varphi(M) \cap (B_N(R) \times \mathbb{R})$ so that the boundary $\partial K \subset \partial B_N(R) \times \mathbb{R}$. Thus we have $(\bar{E}_\kappa - E)|_{\partial K} = 0$ and then $\bar{E}_\kappa \leq E$ in K .

Remark 2.5 When $b < 0$ the equality $\bar{E}_b = E$ in K implies that $|\text{grad}_N s| = |\text{grad}_K s|$. Thus $|\text{grad}_N \rho_N| = |\text{grad}_K \rho_N|$ at every point of K . Recall that $\text{grad}_K \rho_N = \sum_{i=1}^m \langle \text{grad}_N \rho_N, X_i \rangle X_i$. Then $\text{grad}_N \rho_N$ is tangent to K . The integral curves of $\text{grad}_N \rho_N$ are geodesics (liftings of radial geodesics in N via the projection map), in $N \times \mathbb{R}$ and then in K . Thus we conclude that through every point q of K passes a lifting of a radial geodesic of N passing through $\pi_1(q)$. Moreover, going through the computations (2.9) it is easy to see that

$$\sum_{i=2}^m \beta_i^2 = 1 \text{ and } (\gamma_j^i)^2 (\text{Hess}_N \rho_N(\partial/\partial\theta_j, \partial/\partial\theta_j) - \frac{C_b}{S_b}) = 0, \quad i = 2, \dots, m \quad j = 1, \dots, n-1,$$

everywhere in K .

2.4 Proof of Theorem 1.2

We start stating a lemma proved by Palmer.

Lemma 2.6 (Palmer, [21]) Let E_b be the mean time exit of the ball $B_{\mathbb{N}^{m-1}(b)}(R)$. Then

$$E'_b(t) = -\frac{\text{vol}_{m-1}(B_{\mathbb{N}^{m-1}(b)}(t))}{\text{vol}_{m-2}(\partial B_{\mathbb{N}^{m-1}(b)}(t))}. \quad (2.13)$$

Inequality (2.11) says that $\Delta_K \bar{E}_b \circ \varphi \leq -1$ on K . Integration over K yields,

$$-\text{vol}_m(K) = \int_K 1 \geq \int_K -\Delta_K \bar{E}_b \circ \varphi = - \int_{\partial K} \langle E'_b \text{grad } \rho_N \circ \varphi, \nu \rangle \geq - \sup_{\partial K} \|E'_b\| \text{vol}_{m-1}(\partial K)$$

Thus we have that

$$\frac{\text{vol}_{m-1}(\partial K)}{\text{vol}_m(K)} \geq \frac{1}{\sup_{\partial K} \|E'_b\|} = \frac{\text{vol}_{m-2}(\partial B_{\mathbb{N}^{m-1}(b)}(r_K))}{\text{vol}_{m-1}(B_{\mathbb{N}^{m-1}(b)}(r_K))}. \quad (2.14)$$

2.4.1 Mean time exit on spherically symmetric manifolds.

A spherically symmetric manifold is a quotient space $W = ([0, R) \times \mathbb{S}^{n-1}) / \sim$, $R \in (0, \infty]$, where $(t, \theta) \sim (s, \alpha)$ iff $t = s$ and $\theta = \alpha$ or $s = t = 0$, endowed with a Riemannian metric of the form $dt^2 + f^2(t)d\theta^2$, where $f \in C^2([0, R])$ with $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for all $t \in (0, R]$. The class of spherically symmetric manifolds includes the canonical space forms \mathbb{R}^n , $\mathbb{S}^n(1)$ and $\mathbb{H}^n(-1)$. Let $B_W(r) \subset W$ be a geodesic ball of radius r and center $0 = (0 \times \mathbb{S}^{n-1}) / \sim$ in a spherically symmetric manifold $(W, dt^2 + f^2(t)d\theta^2)$. The mean time of the first exit E of $B_W(r)$ is given by

$$E(x) = E(|x|) = - \int_{|x|}^r \frac{1}{f^{n-1}(\sigma)} \int_0^\sigma f^{n-1}(s) ds d\sigma. \quad (2.15)$$

as one can easily check that E satisfies $\Delta_W E = -1$ in $B_W(r)$ with $E|_{\partial B_W(r)} = 0$. Here $|x| = \text{dist}_W(0, x)$. It is also straightforward to show that

$$E'(|x|) = - \frac{\int_0^r f^{n-1}(s) ds}{f^{n-1}(r)} = - \frac{\text{vol}_n(B_W(r))}{\text{vol}_{n-1}(\partial B_W(r))}.$$

Consider the Dirichlet problem $\Delta_W u + \lambda_1(B_W(r))u = 0$ in $B_W(r)$ with $u = 0$ on $\partial B_W(r)$. It was shown by the authors in [5] that the first Dirichlet eigenvalue $\lambda_1(B_W(r))$ is bounded below by

$$\lambda_1(B_W(r)) \geq \frac{[\inf_{B_W(r)} \text{div} X]^2}{4 \sup_{B_W(r)} |X|^2}, \quad (2.16)$$

where X is a vector field in $B_W(r)$ with $\inf \text{div} X > 0$ and $\sup |X| < \infty$. Taking $X = -\text{grad}_W E$ we have that $\text{div} X = 1$ and $|X| = |E'|$. Applying (2.16) we obtain the following theorem.

Theorem 2.7 *Let $B_W(r)$ be a geodesic ball centered at $0 = (0 \times \mathbb{S}^{n-1}) / \sim$ with of radius r in a spherically symmetric Riemannian n -manifold $(W, dt^2 + f^2(t)d\theta^2)$. Let $V(t)$ and $S(t)$ be respectively the n -volume and $(n-1)$ -volume of $B_W(t)$ and $\partial B_W(t)$. Then*

$$\lambda_1(B_W(r)) \geq \inf_{0 \leq t \leq r} \frac{1}{4} \left[\frac{S(t)}{V(t)} \right]^2 \quad (2.17)$$

Corollary 2.8 *Let W be a complete non-compact spherically symmetric manifold. Suppose that the boundary of $B_W(t)$ has volume growth $c_1 e^{c_3 t} \leq S(t) \leq c_2 e^{c_3 t}$, $c_1 < c_2$ and c_3 are positive constants. Then*

$$\lambda^*(W) = \lim_{r \rightarrow \infty} \lambda_1(B_W(r)) \geq \left(\frac{c_1 c_3}{2c_2} \right)^2.$$

Remark 2.9 *The inequality (2.17) should be compared with the inequality $\lambda_1(B_W(r)) \geq \frac{1}{\int_0^r \frac{V(\sigma)}{S(\sigma)} d\sigma}$ proved by Barroso and Bessa in [2].*

2.5 More on submanifolds with tamed second fundamental form

If $\varphi : M \hookrightarrow N$ is a complete m -submanifold with tamed second fundamental form immersed in a Hadamard n -manifold with sectional curvature $K_N \leq b \leq 0$ then φ is proper. Moreover $\varphi(M)$ has finite topology, see [4], [7]. In this section we are going to present the idea to prove that $\varphi(M)$ has finite topology since we need a corollary from its proof. Recall that a submanifold $\varphi : M \hookrightarrow N$ has tamed second fundamental form if $\lim_{i \rightarrow \infty} a_i(M) = a(M) < 1$. Thus given $c \in (a(M), 1)$ there is an $r_o > 0$ such that

$$\|\alpha(\varphi(x))\| \leq c \cdot \frac{C_b}{S_b}(\rho_N(\varphi(x)))$$

for all $x \in M \setminus B_M(r_o)$. Here ρ_N is the intrinsic distance function in N to a point $p = \varphi(q)$. Let $r > r_o$ be such that $\varphi(M) \cap \partial B_N(r)$ and let $\Gamma = \varphi(M) \cap \partial B_N(r)$. Setting $\Lambda = \varphi^{-1}(\Gamma)$ we construct a smooth vector field ν on a open neighborhood of Λ so that $\forall x \in \Lambda$, $y = \varphi(x)$ we have that

$$T_y M = T_y \Gamma \oplus [[d\varphi(x) \cdot \nu(x), \text{grad } \rho_N]]$$

with $\langle d\varphi(x) \cdot \nu(x), \text{grad } \rho_N \rangle > 0$. Here $[[d\varphi(x) \cdot \nu(x), \text{grad } \rho_N]]$ is the vector space generated by $d\varphi(x) \cdot \nu(x)$ and $\text{grad } \rho_N$. For simplicity of notation we are going to identify $d\varphi(x) \cdot \nu(x) = \nu(y)$. Define $\psi(x) = \langle \nu(y), \text{grad } \rho_N(y) \rangle$, $x \in \Lambda$. Since Λ is compact and $\psi(x) > 0$ there is a positive minimum ψ_0 . Consider the Cauchy Problem on M

$$\begin{cases} \xi_t(t, x) &= \frac{1}{\psi} \nu(\xi(t, x)) \\ \xi(0, x) &= x \end{cases} \quad (2.18)$$

It was shown in [7] that ψ satisfies the following differential equation along the integral curves $\xi(t, x)$

$$-(\sqrt{1 - \psi^2})_t = \sqrt{1 - \psi^2} \text{Hess} \rho_N(\omega, \omega) + \langle \nu^*, \alpha(\nu, \nu) \rangle \quad (2.19)$$

Where ν^* is a unit vector normal to ν and ω is a unit normal vector to TM and to $\text{grad } \rho_N$. As consequence of Hessian Comparison Theorem we have the following inequality

$$-(\sqrt{1 - \psi^2})_t \geq \sqrt{1 - \psi^2} \frac{C_b}{S_b} + \langle \nu^*, \alpha(\nu, \nu) \rangle \quad (2.20)$$

It was shown that $\rho_N(\xi(t, x)) = t + r_o$ so we can write $S_b(\rho_N(\xi(t, x))) = S_b(t + r_o)$. The inequality (2.20) is equivalent to

$$\left[S_b(t + r_o) \sqrt{1 - \psi^2} \right]_t \leq -S_b(t + r_o) \langle \nu^*, \alpha(\nu, \nu) \rangle \quad (2.21)$$

Integrating (2.21) of 0 to t we obtain

$$\sqrt{1 - \psi^2}(t) \leq \frac{S_b(r_o)}{S_b(t + r_o)} \sqrt{1 - \psi^2}(0) - \frac{1}{S_b(t + r_o)} \int_0^t S_b(t + r_o) \langle \nu^*, \alpha(\nu, \nu) \rangle ds$$

But $-\langle \nu^*, \alpha(\nu, \nu) \rangle(\xi(s, x)) \leq \|\alpha(\xi(s, x))\| \leq c \cdot (C_b/S_b)(s + r_0)$. Thus

$$\sqrt{1 - \psi^2}(t) \leq \frac{S_b(r_0)}{S_b(t + r_0)} \left[\sqrt{1 - \psi_0^2} - c \right] + c < 1 \quad (2.22)$$

Where $\psi_0 = \inf_{\Lambda} \psi > 0$. Thus for every $x \in \Lambda$ we have that the function ψ satisfies inequality (2.22) along the integral curve $\xi(t, x)$, (which is defined for all t).

Now let $f : M \rightarrow \mathbb{R}$ be defined by $\rho_N \circ \varphi$. The gradient of f is the projection of $\text{grad } \rho_N$ on TM , i.e. $\text{grad } f = \langle \text{grad } \rho_N, \nu \rangle \nu = \psi \cdot \nu$. Set

$$B(b, c, r_0) = \sup_{t, x} \frac{S_b(r_0)}{S_b(t + r_0)} \left[\sqrt{1 - \psi_0^2} - c \right] + c < 1 \quad (2.23)$$

By (2.22) we have that

$$\inf_{M \setminus \varphi^{-1}(B_N(r_0))} \psi \geq \sqrt{1 - B^2(b, c, r_0)} > 0$$

Therefore we have that $\|\text{grad } f(x)\| \geq \sqrt{1 - B^2(b, c, r_0)}$, for $x \in M \setminus \varphi^{-1}(B_N(r_0))$.

2.6 Proof of Theorem 1.5.

Let $\varphi : M \hookrightarrow N$ be a complete m -dimensional submanifold of a complete Hadamard manifold with sectional curvature $b_1 \leq K_N \leq b_2 \leq 0$ with tamed second fundamental form. As we mentioned before φ is proper. Fix a point $p = \varphi(q) \in N$ and $c \in (a(M), 1)$ let $D(R)$ be the extrinsic geodesic ball with center at p and radius $R \geq r_0 = r_0(c) > 0$. Consider $f : M \rightarrow \mathbb{R}$ given by $f = \rho_N^2 \circ \varphi$. We want to estimate $\sup_{D(R)} \Delta f$. We proceed as follows.

$$\begin{aligned} \Delta f &= 2 \sum_i^m \langle X_i, \text{grad } \rho_N \rangle^2 + 2\rho_N \text{Hess } \rho_N(X_i, X_i) + 2\rho_N \langle \text{grad } \rho_N, \vec{H} \rangle \\ &\leq 2 \left[1 + \sup_{t \in [0, R]} \rho_N \frac{C_{b_1}}{S_{b_1}}(\rho_N) \right] + \max\{2r_0 \sup_{B_N(r_0)} |H|, 2c \cdot R \cdot (C_{b_2}/S_{b_2})(R)\} \\ &= 2 \left[1 + \sup_{t \in [0, R]} \rho_N \frac{C_{b_1}}{S_{b_1}}(\rho_N) \right] + \Lambda(\sup_{B_N(r_0)} |H|, c, r_0, R, b_2) \end{aligned} \quad (2.24)$$

On the other hand $\text{grad } f = 2\rho_N \psi \nu$. Thus we have that

$$\inf_{\partial D(R)} \|\text{grad } f\| \geq 2 \cdot R \cdot \sqrt{1 - B^2(b_2, c, r_0)}$$

By Green's Theorem we have that

$$\begin{aligned} \sup_{D(R)} \Delta f \cdot \text{vol}_m(D(R)) &\geq \int_{D(R)} \Delta f = \int_{\partial D(R)} \langle \text{grad } f, \nu \rangle \\ &\geq \inf_{\partial D(R)} \|\text{grad } f\| \cdot \text{vol}_{m-1}(\partial D(R)) \end{aligned}$$

From we obtain

$$\frac{\text{vol}_{m-1}(\partial D(R))}{\text{vol}_m(D(R))} \leq \frac{1 + \sqrt{-b_1} \cdot R \cdot \coth(\sqrt{-b_1} \cdot R) + \Lambda(\sup_{B_N(r_0)} |H|, c, r_0, R, b_2)}{R \cdot \sqrt{1 - B^2(b_2, c, r_0)}}$$

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